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RING THEORY-SEM-IV

INTRODUCTION TO RINGS, SUBRINGS, IDEALS, HOMOMORPHISM, POLYNOMIAL RINGS

UNIT-2: SUBRINGS AND IDEALS

Subring: Let $(R;\ +,\cdot\ )$ be ring and S be a non-empty subset of R. If $(S;\ +,\cdot\ )$ is ring w.r.to the two operation in R then we say that S is a subring of R.

Ex: 1. $(\mathbb{Z};\ +,\cdot\ )\ \ (\mathbb{Q};\ +,\cdot\ )\ (\mathbb{R};\ +,\cdot\ )\ are\ all\ subrings\ of\ \ (\mathbb{C};\ +,\cdot\ )$

2. Let $S = \left\{\frac{a}{2}/a \in \mathbb{Z}\right\}$ is not a subring of $(\mathbb{Q};\ +,\cdot\ )\ because\ \frac{1}{2},\frac{1}{2} \in S \Longrightarrow \frac{1}{4} \notin S$

$\therefore\ Multiplication\ is\ not\ a\ binary\ operation.$

Remark: 1. Every ring has at least two subrings. They are S= {0} and S=R these are called trivial subrings. If any other than these subrings exist then it is called as non-trivial subrings.

Theorem1: Let S be a non-empty subset of a ring $(R;\ +,\cdot\ )$ then

S is a subring of R $\Leftrightarrow$ $\boldsymbol{(i) a \in S, b \in S \Longrightarrow a - b \in S\ \ (ii) a \in S, b \in S \Longrightarrow a.b \in S\ \ \ \forall a, b \in S}$.

Proof: Necessary condition ($\Longrightarrow$): Given that S is a subring

To prove that $(i) a \in S, b \in S \Longrightarrow a - b \in S\ \ (ii) a \in S, b \in S \Longrightarrow a.b \in S\ \ \ \forall a, b \in S$

since S is a subring $\Longrightarrow$ S is a ring under the same operation of R

Now $(i) a \in S, b \in S \Longrightarrow a \in S, -b \in S \Longrightarrow a + (-b) \in S\ \Longrightarrow a - b \in S$

$(ii) a \in S, b \in S \Longrightarrow a.b \in S\ \ \ \forall a, b \in S$

Sufficient condition ($\Longleftarrow$): Conversely given that S is a non-empty subset of a ring $(R;\ +,\cdot\ )$ such that $(i) a \in S, b \in S \Longrightarrow a - b \in S\ \ (ii) a \in S, b \in S \Longrightarrow a.b \in S\ \ \ \forall a, b \in S$

To prove that S is a subring

(i) Since S is non-empty. Let $a \in S$

By $(i) a \in S, a \in S \Longrightarrow a - a \in S \Longrightarrow 0\ \in S\ so\ 0\ is\ the\ zero\ element.$

(ii) $By\ (i) 0 \in S, a \in S \Longrightarrow 0 - a \in S \Longrightarrow -a\ \in S \Longrightarrow -a\ is\ additive\ inverse\ of\ 'a'$

(iii) $Let\ a \in S, b \in S \Longrightarrow a \in S, -b \in S\ \Longrightarrow a - (-b) \in S\ \Longrightarrow a + b \in S\ \ so' +'$ is binary on S

(iv) Since all the elements of S are in R ($\subseteq R$) since the operation $+$ is commutative, associative in R. $\therefore$ The operation $+$ is commutative, associative in S

(v) By con (ii) $'\ \cdot '$ is binary operation in S

(vi) Let $a, b, c \in S \Longrightarrow a, b, c \in R \Longrightarrow (ab)c = a(bc) \Longrightarrow '\cdot 'is\ associative\ in\ S$

(vii) Let $a, b, c \in S \Longrightarrow a, b, c \in R \Longrightarrow a.(b+c) = a.b + a.c \;\; and \;\; (b+c).a = b.a + c.a$

$' \cdot '$ is distributive under addition in S Hence $(S; +, \cdot)$ $is\ a\ ring$

Theorem2: The intersection of two subrings of a ring R is a subring of R

Proof: Let S_1, S_2 be two subrings of a ring R. To prove that $S_1 \cap S_2$ is subring of R

Since Every subring contains at least zero element of the ring. So $0 \in S_1, 0 \in S_2 \Longrightarrow 0 \in S_1 \cap S_2$

$\Longrightarrow S_1 \cap S_2 \neq \phi$ and $S_1 \cap S_2 \subseteq R$

Let $a, b \in S_1 \cap S_2 \Longrightarrow a, b \in S_1 \;\; and\ a, b \in S_2$

Since $a, b \in S_1 \;\; and \;\; S_1\ is\ a\ subring\ of\ R \Longrightarrow a - b \in S_1 \;\; and\ a.b \in S_1 \rightarrow (1)$

Also $a, b \in S_2 \;\; and \;\; S_2\ is\ a\ subring\ of\ R \Longrightarrow a - b \in S_2 \;\; and\ a.b \in S_2 \rightarrow (2)$

From (1) and (2) $a - b \in S_1 \cap S_2\ and \;\; ab \in S_1 \cap S_2$

$a, b \in S_1 \cap S_2 \Longrightarrow a - b \in S_1 \cap S_2\ and \;\; ab \in S_1 \cap S_2. hence\ S_1 \cap S_2$ is subring of R

Theorem3: If S_1 and S_2 are subrings of R then $S_1 \cup S_2$ is subring of $R \Longleftrightarrow S_1 \subseteq S_2 or\ S_2 \subseteq S_1$

Proof: Necessary condition ($\Longrightarrow$): Given that $S_1 \cup S_2$ is subring of R.

To prove that $S_1 \subseteq S_2 or\ S_2 \subseteq S_1$

If possible suppose that $S_1 \nsubseteq S_2\ and \;\; S_2 \nsubseteq S_1$

Since $S_1 \nsubseteq S_2 \Longrightarrow \exists\ an\ element\ a \in S_1\ and \;\; a \notin S_2$

$also \;\; S_2 \nsubseteq S_1 \Longrightarrow \exists\ an\ element\ b \in S_2\ and \;\; b \notin S_1$

$\therefore \;\; a, b \in S_1 \cup S_2 \Longrightarrow a + b \in S_1 \cup S_2 \Longrightarrow a + b \in S_1\ or\ a + b \in S_2$

If $a + b \in S_1 :-$

$a + b \in S_1, a \in S_1 \;\; and \;\; S_1\ is\ a\ subring \Longrightarrow a + b - a \in S_1 \Longrightarrow b \in S_1\ which\ is\ absurd.$

If $a + b \in S_2 :-$

$a + b \in S_2, b \in S_2 \;\; and \;\; S_2\ is\ a\ subring \Longrightarrow a + b - b \in S_2 \Longrightarrow a \in S_2\ which\ is\ absurd$

$\therefore our\ supposition\ is\ a\ wrong. Hence \;\; S_1 \subseteq S_2 or\ S_2 \subseteq S_1$

Sufficient condition ($\Longleftarrow$): Given S_1, S_2 are the subrings of R such that $S_1 \subseteq S_2 or\ S_2 \subseteq S_1$

To prove that $S_1 \cup S_2$ is subring of R

$S_1 \subseteq S_2 \Longrightarrow S_1 \cup S_2 = S_2 \Longrightarrow S_1 \cup S_2$ is subring of R [since $S_2$is a subring of R]

$S_2 \subseteq S_1 \Longrightarrow S_1 \cup S_2 = S_1 \Longrightarrow S_1 \cup S_2$ is subring of R [since $S_1$is a subring of R]

IDEALS

Ideal: Let $(R; +,\cdot)$ be a ring. A non-empty subset I of R is called an ideal of R. if

(i) $a \in I, b \in I \Longrightarrow a - b \in I$ $\quad$ $(ii) a \in I,\ r \in R \Longrightarrow ar \in I(\ Right\ \ ideal),$

$ra \in I\ (left\ ideal)$

Ex: The set of even integers is an ideal of the ring of integers.

Remark:1. Every ideal I of R is subring of R but a subring need not be an ideal. For example the $(\mathbb{Z}; +,\cdot)$ is a subring of $(\mathbb{R}; +,\cdot)$ but $\mathbb{Z}$ is not an ideal of $\mathbb{R}$ because 1$\in \mathbb{Z}, \frac{3}{4} \in \mathbb{R}$

$\Longrightarrow 1.\frac{3}{4} = \frac{3}{4} \notin \mathbb{Z} \qquad \therefore \mathbb{Z}\ is\ not\ an\ ideal\ of\ \mathbb{R}$

2. $I = \{0\}\ is\ called\ as\ null\ ideal\ of\ R\ and\ I = R\ is\ called\ as\ unit\ ideal\ of\ R.$

$These\ two\ ideals\ are\ called\ trivial\ ideals\ of\ R\ and\ any\ other\ than\ these\ ideals\ exist\ then$

$it\ is\ called\ as\ non-trivial\ ideals.$

Theorem1: The intersection of two ideals of a ring R is an ideal of R

Proof: Let I_1, I_2 be two ideals of a ring R. To prove that $I_1 \cap I_2$ is ideal of R

Since every ideal contains at least zero element of the ring. So $0 \in I_1, 0 \in I_2 \Longrightarrow 0 \in I_1 \cap I_2$

$\Longrightarrow I_1 \cap I_2 \neq \phi$ and $I_1 \cap I_2 \subseteq R$

(i) Let $a, b \in I_1 \cap I_2 \Longrightarrow a, b \in I_1\ \ and\ a, b \in I_2$

Since $a, b \in I_1\ \ and\ \ I_1\ is\ an$ ideal $of\ R \Longrightarrow a - b \in I_1$

Also $a, b \in I_2\ \ and\ \ I_2\ is\ an$ ideal $of\ R \Longrightarrow a - b \in I_2$

$\therefore\ \ a - b \in I_1 \cap I_2$

(ii) Let $a \in I_1 \cap I_2, and\ r \in R\ then\ a \in I_1, and\ a \in I_2$

Now $a \in I_1, r \in R\ and\ I_1\ is\ an\ ideal\ of\ R \Longrightarrow ar, ra \in I_1 \rightarrow (1)$

$a \in I_2, r \in R\ and\ I_2\ is\ an\ ideal\ of\ R \Longrightarrow ar, ra \in I_2 \to (2)$

From (1) and (2) $ar, ra \in I_1 \cap I_2$

Thus $a \in I_1 \cap I_2, and\ r \in R\ then\ ar, ra \in I_1 \cap I_2$ $\quad Hence\ I_1 \cap I_2$ is ideal of R

Theorem3: If $I_1\ and\ I_2$ are ideals of R then $I_1 \cup I_2$ is an ideal of $R \Longleftrightarrow I_1 \subseteq I_2 or\ I_2 \subseteq I_1$

Proof: Necessary condition ($\Longrightarrow$): Given that $I_1 \cup I_2$is an ideal of R.

To prove that $I_1 \subseteq I_2 or\ I_2 \subseteq I_1$

If possible suppose that $I_1 \nsubseteq I_2\ and\ I_2 \nsubseteq I_1$

Since $I_1 \nsubseteq I_2 \Longrightarrow \exists\ an\ element\ a \in I_1\ and\ a \notin I_2$

$also\ \ I_2 \nsubseteq I_1 \Longrightarrow \exists\ an\ element\ b \in I_2\ and\ b \notin I_1$

$\therefore\ a, b \in I_1 \cup I_2 \Longrightarrow a - b \in I_1 \cup I_2 \Longrightarrow a - b \in I_1\ or\ a - b \in I_2$

If $a - b \in I_1 : -$

$a \in I_1, a - b \in I_1, and\ I_1\ is\ a\ ideal \Longrightarrow a - (a - b) \in I_1 \Longrightarrow b \in I_1\ which\ is\ absurd.$

If $a - b \in I_2 : -$

$a - b \in I_2, b \in I_2\ and\ I_2\ is\ an\ ideal\ of\ R \Longrightarrow a - b + b \in I_2 \Longrightarrow a \in I_2\ which\ is\ absurd$

$\therefore our\ supposition\ is\ a\ wrong. Hence\ I_1 \subseteq I_2 or\ I_2 \subseteq I_1$

Sufficient condition ($\Longleftarrow$): Given I_1, I_2 are the ideals of R such that $I_1 \subseteq I_2 or\ I_2 \subseteq I_1$

To prove that $I_1 \cup I_2$ is an ideal of R

$I_1 \subseteq I_2 \Longrightarrow I_1 \cup I_2 = I_2 \Longrightarrow I_1 \cup I_2$ is an ideal of R [since $I_2$is an ideal of R]

$I_2 \subseteq I_1 \Longrightarrow I_1 \cup I_2 = I_1 \Longrightarrow I_1 \cup I_2$is an ideal of R [since $I_1$is an ideal of R]

Theorem: 3 If $I_1\ and\ I_2$ are ideals of R then $I_1 + I_2$ is an ideal of R

Proof: Given $I_1\ and\ I_2$ are ideals of R. To prove that $I_1 + I_2$ is an ideal of R

Since every ideal contains at least zero element of the ring.

(i) So $0 \in I_1, 0 \in I_2 \Longrightarrow 0 + 0 = 0 \in I_1 + I_2 \Longrightarrow I_1 + I_2 \neq \phi$

(ii) Let $x \in I_1 + I_2\ then\ x = x_1 + x_2\ where\ x_1 \in I_1\ and\ x_2 \in I_2$

$x_1 \in I_1 \Longrightarrow x_1 \in R\ (\because I_1 \subseteq R)\ , also\ x_2 \in I_2 \Longrightarrow x_2 \in R\ (\because I_2 \subseteq R)\ ,$

$\therefore\ x_1 + x_2 \in R \qquad Thus\ I_1 + I_2 \subseteq R$

(iii) Let $x, y \in I_1 + I_2\ then x = x_1 + x_2\ , y = y_1 + y_2 where\ x_1, y_1 \in I_1\ and\ \ x_2, y_2 \in I_2$

Now $-y = (x_1 + x_2) - (y_1 + y_2) = (x_1 - y_1) + (x_2 - y_2) \in I_1 + I_2\ [\because x_1 - y_1 \in I_1\ and$
$x_2 - y_2 \in I_2]$

(iv) Let $x \in I_1 + I_2\ \ and\ \ r \in R\ then\ x = x_1 + x_2\ where\ x_1 \in I_1\ and\ \ x_2 \in I_2$

Now $rx = r(x_1 + x_2) = rx_1 + rx_2 \in I_1 + I_2\ [\because rx_1 \in I_1\ and\ \ rx_2 \in I_2\ ]$

Also $xr = (x_1 + x_2)r = x_1r + x_2r \in I_1 + I_2\ \ [\because x_1r \in I_1\ and\ \ x_2r \in I_2]$

$\therefore\ x \in I_1 + I_2, \quad r \in R\ \Longrightarrow rx, xr \in I_1 + I_2$

$Thus\ I_1 + I_2\ is\ an\ ideal\ of\ R$

Theorem:4 If I is an ideal of a ring R with unity and 1$\in$ $I\ then\ I = R$

Proof: Since I is an ideal of a ring R with unity. By the definition of an ideal of $I \subseteq R \longrightarrow (1)$

Let x be any element of R

$\therefore\ x \in R \Longrightarrow x.1 \in R \Longrightarrow x.1 \in I\ \ [\because 1 \in I\ \ , x \in R\ and\ I\ is\ an\ ideal\ of\ R \Longrightarrow x.1 \in I\ \ ]$

$\Longrightarrow x \in I \qquad \therefore\ R \subseteq I \longrightarrow (2)$

From (1) &(2) $I = R$

Theorem5: A field has no proper ideals (OR) Every ideal of a field F has only {0} and F itself

Proof: Let I be an ideal of a ring F such that $I \neq \{0\}$

Now we prove that $I = F$ By the definition of an ideal of $I \subseteq F \longrightarrow (1)$

Let $a \in I\ so\ that\ a \neq 0.$

let $a \neq 0 \in F\ \ and\ F\ is\ a\ field\ \ \Longrightarrow \exists a^{-1} \in F \ni aa^{-1} = a^{-1}a = 1$

$\therefore\ a \in I,\ a^{-1} \in F,\ and\ I\ is\ an\ ideal \Longrightarrow aa^{-1} \in I \Longrightarrow 1 \in I$

Let x be any element of F

$\therefore\ x \in F \Longrightarrow x.1 \in F \Longrightarrow x.1 \in I\ \ [\because 1 \in I\ \ , x \in F\ and\ I\ is\ an\ ideal\ of\ F \Longrightarrow x.1 \in I\ \ ]$

$$\Longrightarrow x \in I \qquad \therefore\ F \subseteq I \longrightarrow (2)$$

From (1) &(2) $I = F$

$\therefore$ A field F has no proper ideals

Problems

1. $\boldsymbol{Show\ that\ S = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} / a, b, c \in \mathbb{Z}\right\}\ is\ a\ Sub\ ring\ of\ the\ ring\ of\ 2 \times 2}$

$\boldsymbol{matrices\ whose\ elements\ are\ integers.}$

Sol: $Let\ R = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in \mathbb{Z}\right\}\ be\ the\ ring\ under\ addition\ and\ multiplications$

$of\ matrices. Clearly \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\ is\ the\ zero\ element\ of R.$

$Given\ S = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} / a, b, c \in \mathbb{Z}\right\}$

$To\ prove\ that\ S\ is\ Sub\ ring\ of\ R$

$(i)\ Clearly\ S \neq \emptyset\ and\ S \subseteq R$

$(ii) Let\ A, B \in S\ so\ that\ A = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix};\ \ B = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} where\ a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{Z}$

$Now\ A - B = \begin{bmatrix} a_1 - a_2 & b_1 - b_2 \\ 0 & c_1 - c_2 \end{bmatrix} \in S\ since\ a_1 - a_2, b_1 - b_2, c_1 - c_2 \in \mathbb{Z}$

$Also\ AB = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{bmatrix} \in S\ since\ a_1a_2, a_1b_2 + b_1c_2, c_1c_2 \in \mathbb{Z}$

$\therefore\ A, B \in S\ \Longrightarrow A - B \in S\,; A \cdot B \in S\ \Longrightarrow S\ is\ Sub\ ring\ of\ R$

2. $\boldsymbol{Show\ that\ I = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} / a, b \in \mathbb{Z}\right\}\ is\ a\ \ right\ ideal\ but\ not\ left}$

$\boldsymbol{ideal\ of\ the\ ring\ of\ 2 \times 2\ matrices\ whose\ elements\ are\ integers.}$

$\boldsymbol{(OR)}$

$\boldsymbol{Show\ that\ I = \left\{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} / a, b \in \mathbb{Z}\right\}\ is\ a\ \ right\ ideal\ but\ not\ left}$

$\boldsymbol{ideal\ of\ the\ ring\ of\ 2 \times 2\ matrices\ whose\ elements\ are\ integers}$

Sol: $Given\ \ I = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} / a\,, b \in \mathbb{Z}\right\}$

$Let\ R\ be\ the\ \ set\ of\ all\ 2 \times 2\ matrices\ whose\ elements\ are\ integers.$

$\therefore (R, +, \cdot)\ is\ ring\ under\ addition\ and\ multiplications\ of\ matrices.$

$(i)\ Clearly\ I \neq \emptyset \left(\because \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in I\right)\ \ and\ I \subseteq R.$

$(ii)\ Let\ A, B \in I\ so\ that\ A = \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix};\ \ B = \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} where\ a_1, b_1, a_2, b_2 \in \mathbb{Z}$

$Now\ A - B = \begin{bmatrix} a_1 - a_2 & b_1 - b_2 \\ 0 & 0 \end{bmatrix} \in I\ since\ a_1 - a_2, b_1 - b_2 \in \mathbb{Z}$

$Let\ X = \begin{bmatrix} l & m \\ n & p \end{bmatrix} \in R\ where\ l, m, n, p \in \mathbb{Z}$

$Now\ \ AX = \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} l & m \\ n & p \end{bmatrix} = \begin{bmatrix} a_1 l + b_1 n & a_1 m + b_1 p \\ 0 & 0 \end{bmatrix} \in I$

$since\ a_1 l + b_1 n, a_1 m + b_1 p \in \mathbb{Z}$

$\therefore\ \ I\ is\ a\ right\ ideal\ of\ R$

$but\ we\ have\ A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in I\ and\ X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\ \in R$

$\therefore XA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin I$

$\therefore\ \ I\ is\ not\ a\ left\ ideal\ of\ R$

3. Show that $I = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} / a\,, b \in \mathbb{Z}\right\}$ is a left ideal but not right ideal of the ring of 2×2 matrices whose elements are integers.

(OR)

Show that $I = \left\{\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} / a\,, b \in \mathbb{Z}\right\}$ is a left ideal but not right ideal of the ring of 2×2 matrices whose elements are integers

Sol: $Given\ \ I = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} / a\,, b \in \mathbb{Z}\right\}$

$Let\ R\ be\ the\ \ set\ of\ all\ 2 \times 2\ matrices\ whose\ elements\ are\ integers.$

$\therefore (R,+,\cdot)$ *is ring under addition and multiplications of matrices.*

(i) *Clearly* $I \neq \emptyset \left(\because \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in I\right)$ *and* $I \subseteq R$.

(ii) *Let* $A, B \in I$ *so that* $A = \begin{bmatrix} a_1 & 0 \\ b_1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} a_2 & 0 \\ b_2 & 0 \end{bmatrix}$ *where* $a_1, b_1, a_2, b_2 \in \mathbb{Z}$

Now $A - B = \begin{bmatrix} a_1 - a_2 & 0 \\ b_1 - b_2 & 0 \end{bmatrix} \in I$ *since* $a_1 - a_2, b_1 - b_2 \in \mathbb{Z}$

Let $X = \begin{bmatrix} l & m \\ n & p \end{bmatrix} \in R$ *where* $l, m, n, p \in \mathbb{Z}$

Now $XA = \begin{bmatrix} l & m \\ n & p \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ b_1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 l + m b_1 & 0 \\ a_1 n + b_1 p & 0 \end{bmatrix} \in I$

since $a_1 l + m b_1, a_1 n + b_1 p \in \mathbb{Z}$

$\therefore$ *I is a left ideal of R*

but we have $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in I$ *and* $X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in R$

$\therefore AX = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin I$

$\therefore$ *I is not a right ideal of R*

4. *Show that* $S = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} / a, b \in \mathbb{Z}\right\}$ ***is a Sub ring of the ring of 2×2 matrices whose elements are integers but which neither left ideal nor right ideal .***

(OR)

Show that $S = \left\{\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} / a, b \in \mathbb{Z}\right\}$ ***is a Sub ring of the ring of 2×2 matrices whose elements are integers but which neither left ideal nor right ideal .***

Sol: *Given* $S = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} / a, b \in \mathbb{Z}\right\}$

Let R be the set of all 2×2 *matrices whose elements are integers.*

$\therefore (R, +, \cdot)$ *is ring under addition and multiplications of matrices.*

(i) *Clearly* $S \neq \emptyset \left(\because \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S\right)$ *and* $S \subseteq R$.

(ii) *Let* $A, B \in I$ *so that* $A = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$; $B = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$ *where* $a_1, b_1, a_2, b_2 \in \mathbb{Z}$

Now $A - B = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ 0 & b_1 b_2 \end{bmatrix} \in S$ *since* $a_1 a_2, b_1 b_2 \in \mathbb{Z}$

Also $AB = \begin{bmatrix} a_1 - a_2 & 0 \\ 0 & b_1 - b_2 \end{bmatrix} \in S$ *since* $a_1 - a_2, b_1 - b_2 \in \mathbb{Z}$

$\therefore\ A, B \in S \implies A - B \in S\,; A \cdot B \in S \implies S$ *is Sub ring of* R

But for $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S,\qquad X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in R$

Now $XA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin S$

$\therefore\ S$ *is not a left ideal of* R

Also $AX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin S$

$\therefore\ S$ *is not a right ideal of* R.

Hence S is neither left ideal nor right ideal